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SOME PROBLEMS IN THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS AND THEIR NONLOCAL ANALOGUES

ABSTRACT

of the PhD thesis for the degree of doctor of Philosophy (PhD) in the specialty «8D05404-Fundamental and applied mathematics»

The relevance of the research topic. Fractional order calculations are widely used in mathematical modeling, mechanics, physics, etc. In recent years, the study of linear and quasi-partial diffusion equations containing fractional derivatives has attracted the attention of many scientists, in particular L. Caffarelli, M. Yamamoto, R. Zacher, Y. Luchko, S. Dipiero, E. Valdinocci, M. Kirane, M. Ruzhansky, B. Turmetov and others. Accordingly, when studying the initial-boundary value problems for these equations, the results on their local and global resolutions are quite relevant.

Apart from the well-posedness of the evolutionary equation problems, the asymptotic features of the solutions are particular interesting. Asymptotic estimates of solutions to parabolic problems with a constant coefficient or fractional order with a variable coefficient are obtained in the works of Y. Luchko, M. Yamamoto, Ch. Lee, R. Zacher, E. Valdinocci, etc.

Asymptotic estimates of solutions to the subdiffusion equation with variable coefficients have also been sufficiently studied. As far as we are aware, while certain works by B. Turmetov and B. Kadirkulov, H. Dong and D. Kim, R. Sacher et al. have addressed the solution of initial boundary value problems for fractional order diffusion equations with a singular coefficient, asymptotic estimates of solutions have not been examined.

Fractional differential equations with time-dependent coefficients simulate the processes of anomalous diffusion.

The aim of the PhD thesis is to study the following fractional differential equations

$$\partial_t^\alpha u(x,t) - a(t)A\big(u(x,t)\big) = 0, \qquad (x,t) \in \Omega \times \mathbb{R}_+$$

where $0 < \alpha \leq 1$, $a(t) \in L^1_{loc}(\mathbb{R}_+)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, or $\Omega \equiv \mathbb{R}^N$ and ∂_t^α – is one of the next fractional order operators:

- The Caputo fractional derivative:

$$\partial_t^\alpha = {}^cD_{0+,t}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \partial_s u(x,s) ds;$$

- The Riemann-Liouville fractional derivative:

$$\partial_t^\alpha = D_{0+,t}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} u(x,s) ds;$$

- The Caputo – Hadamard fractional derivative:

$$\partial_t^\alpha = {}_H^C D_{a+,t}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(log \frac{t}{s}\right)^{-\alpha} f'(s) \frac{ds}{s};$$

- The Hadamard fractional derivative:

$$\partial_t^\alpha = {}_H D_{a+,t}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} t \frac{d}{dt} \int_a^t \left(log \frac{t}{s}\right)^{-\alpha} f(s) \frac{ds}{s}$$

and $A(u)$ one of the following linear and quasi-linear operators:

- Laplace operator:

$$A(u) \coloneqq \Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2};$$

- p- Laplace operator:

$$A(u) \coloneqq \Delta_p u = div(|\nabla u|^{p-2} \nabla u);$$

- Porous medium operator:

$$A(u) \coloneqq \nabla(g(u) \nabla u);$$

- Degenerate elliptic operator:

$$A(u) \coloneqq f(u)\Delta u;$$

- Mean curvature operator:

$$A(u) \coloneqq div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right);$$

- Kirchhoff operator:

$$A(u) \coloneqq M(\|\nabla u\|_{L^q})\Delta u.$$

Moreover, we assume that the function $a(t)$ satisfies the following inequality

$$(H): a(t) \geq kt^{\beta}, \beta > -\alpha, k > 0.$$

Object of research.

- The solvability of initial-boundary value and initial value problems for degenerate diffusion equations involving the time fractional derivatives of the Riemann-Liouville, Caputo, Hadamard, Caputo-Hadamard is investigated;
- Asymptotic estimates of solutions of linear and quasi-partial parabolic problems with a variable coefficient involving the Caputo fractional derivative in time are obtained;
- Analogs of the Leibniz rule were derived for Hadamard and Caputo-Hadamard derivatives, and their application in obtaining a priori estimates of solutions to initial-boundary value problems posed in fractional order diffusion equations is given.

The methods of scientific research. The dissertation tasks have been studied using techniques and methods from the theory of partial differential equations, functional analysis, and fractional calculations.

Scientific novelty of the work. This dissertation work investigates new problems of linear and quasilinear differential equations involving fractional order derivatives. it also includes classical linear problems, as well as a class of quasi-particle problems with important applications. The majorities of the topics under discussion have never been examined before or were only addressed with regard to certain cases. Therefore, research work generalizes known results and includes completely new ones.

Theoretical and practical significance of the results. The subject of the study is mainly theoretical and fundamental. However, the problems considered in the dissertation have application in the description of anomalous diffusion processes.

Publications. On the topic of the thesis 13 papers were published, including 2 publications in a high-ranking scientific journals, indexed in the Web of Science and Scopus, 3 publications in scientific journals included in the list recommended by the Committee for Quality Assurance in the Field of Science and Higher Education of the Ministry of Science and Higher Education of the Republic of Kazakhstan for publication of the main scientific results of scientific activities, 1 paper was accepted for foreign publication and 7 publications in materials of foreign international conferences**.**

The results on the topic of the thesis were published in the following papers:

Publication in the high-ranking scientific journals

1. A. G. Smadiyeva, B. T. Torebek, Decay estimates for the time-fractional evolution equations with time-dependent coefficients // Physical and Engineering Sciences. — 2023. — 479:2276— 20230103, **Scopus (SJR 0.8, Quartile 1, Percentile 95).**

2. A. G. Smadiyeva, Degenerate time-fractional diffusion equation with initial and initial-boundary conditions // Georgian Mathematical Journal. — 2023. — Vol. 30. — P. 435–443, **Scopus (SJR 0.3, Quartile 3, Percentile 50).**

Acceptted for publication

1. A. G. Smadiyeva, Degenerate Diffusion Equation with the Hadamard Time-Fractional Derivative // Women in Analysis and PDE: Research Perspectives Ghent Analysis and PDE Center. —2024, **(Scopus). https://doi.org/10.1007/978-3-031-57005-6-38. Acceptted**

In the journals recommended by the quality assurance committee in the field of science and higher education of R&D of the Republic of Kazakhstan:

1. M. Borikhanov, A. G. Smadiyeva, Cauchy problems for the time-fractional degenerate diffusion equations // Journal of Mathematics, Mechanics and Computer Science. — 2023. — Vol. 117, no. 2. — Pp. 15–23.

2. A. G. Smadiyeva, Well-posedness of the initial-boundary value problems for the time-fractional degenerate diffusion equations // Bulletin of the Karaganda University, Mathematics Series. — 2022.— Vol. 107. — Pp. 145–151.

3. A. Smadiyeva, Initial-boundary value problem for the time-fractional degenerate diffusion equation//Journal of Mathematics, Mechanics and Computer Science. — 2022. — Vol. 113, no. 1. — Pp. 32–41.

Publications in materials of international conferences

1. A. Smadiyeva, Some problems for the degenerate subdiffusion equation // VI International Conference, Nalchik, December 5-9, 2021, Pp. 176.

2. A. Smadiyeva, On an integro-differential diffusion equation // IX International Conference, Aktobe, May 24-28, 2022.

3. Smadiyeva Asselya, On an initial-boundary-value problems for the time-fractional diffusion equation// XII INTERNATIONAL CONFERENCE OF THE GEORGIAN MATHEMATICAL UNION, Batumi, August 29 - September 3. 2022

4. A. G. Smadiyeva. The solvability of degenerate diffusion equation with the Hadamard time-fractional derivative // International Conference "Analysis, Differential Equations and their Applications", Astana, Kazakhstan, June 21-24, 2023.- p. 152.

5. Asselya Smadiyeva. Decay estimates for the degenerate time-fractional evolution equations//VII World Congress of Mathematicians of the Turkic World (TWMS Congress-2023), Turkestan, Kazakhstan, 20-23 September 2023.

6. Asselya Smadiyeva. A Cauchy-Dirichlet problem for the Hadamard time-fractional diffusion equation //PROGRAM OF THE SUMMER SCHOOL WORKSHOP “Modern Problems in PDEs and Applications” Ghent Analysis & PDE Center, Ghent University, Ghent, Belgium, 11 August 2023.

7. A. G. Smadiyeva, Decay estimates for Cauchy-Dirichlet problems // Annual International April Mathematical Conference, Almaty, April, 2024, p. 145.

The structure and scope of the thesis. The PhD thesis includes a title page, content, introduction, four chapters, conclusion and list of references, consisting of 52 titles. The total volume of the thesis is 107 pages.

The main content of the thesis.

In Chapter 1, the definitions, lemmas and properties used in the dissertation work were considered, and the Cauchy, Cauchy-Dirichlet, Cauchy-Neumann problems posed in the Riemann-Liouville fractional order diffusion equation. The existence, convergence and uniquness of solutions to the problems presented are proved.

In Chapter 2, the solvability of the problems and the estimates of the solutions for linear and quasilinear diffusion equations with a fractional derivative in the Caputo sense are investigated.

In Chapter 3 and Chapter 4, linear and quasilinear diffusion equations with the Hadamard and Caputo-Hadamard fractional derivatives are considered. Aditionally, Leibniz's rule analogs for derivatives of the fractional order Hadamard and Caputo-Hadamard were also obtained, and their application to a priori estimates of problem solutions for linear, quasilinear fractional order diffusion equations was examined.